



TITLE:

FitzHugh-Nagumo方程式に現れる 微細パターンについて (反応拡散系 におけるパターン形成と漸近的幾 何構造の研究)

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CITATION:

Oshita, Yoshihito. FitzHugh-Nagumo方程式に現れる微細パターンについて (反応拡散系におけるパターン形成と漸近的幾何構造の研究). 数理解析研究所講究録 2004, 1356: 82-90

ISSUE DATE:

2004-02

URL:

<http://hdl.handle.net/2433/25186>

RIGHT:

FitzHugh–Nagumo 方程式に現れる微細パターンについて

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1 Introduction

FitzHugh–Nagumo equation was introduced as a reduced equation of Hodgkin–Huxley model, which describes propagation of signals along a nerve axon. It has turned out to be related to the theory of the pattern formation in mathematical biology and wave propagation in excitable media. Refer to [2, 3, 5, 6, 7, 8]. FitzHugh–Nagumo equation is a system of reaction–diffusion equation consisting of two unknown functions u and v representing concentrations of activator and inhibitor respectively, and typically of the form

$$(E-1)_\varepsilon \quad \begin{aligned} u_t &= \varepsilon^2 \Delta u + f(u) - \kappa v, \\ \tau v_t &= D \Delta v + u - m - \gamma v, \end{aligned} \quad \text{in } \Omega \times \mathbb{R}_+$$

with the homogeneous Neumann boundary condition on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain; $f(u) = -W'(u)$ ($W \in C^2(\mathbb{R})$ is a double-well potential which has global minima exactly at ± 1 , and $W(\pm 1) = 0$) is a bistable nonlinearity; $m \in (-1, +1)$ is a constant; κ, τ, D and γ are positive constants and ε is a positive parameter. Throughout this survey we always impose the homogeneous Neumann boundary condition. We study the parameter scaling $\varepsilon \rightarrow 0$ in $(E-1)_\varepsilon$. We also study the following scaling.

$$(E-2)_\varepsilon \quad \begin{aligned} u_t &= \varepsilon^2 \Delta u + f(u) - \frac{\varepsilon}{\mu} v, \\ \tau v_t &= D \Delta v + u - m - \gamma v, \end{aligned} \quad \text{in } \Omega \times \mathbb{R}_+$$

where μ, τ, D and γ are positive constants and $\varepsilon (\rightarrow 0)$ is a positive parameter. In addition, we study another scaling, that is,

$$(E-3)_{\varepsilon, D} \quad \begin{aligned} u_t &= \varepsilon^2 \Delta u + f(u) - \frac{\varepsilon}{\mu} v, \\ \tau v_t &= D \Delta v + u - m - \gamma v, \end{aligned} \quad \text{in } \Omega \times \mathbb{R}_+$$

where μ, τ and γ are positive constants and $\varepsilon(\rightarrow 0)$ and $D(\rightarrow \infty)$ are positive parameters. Stationary solutions of $(E-1)_\varepsilon$ are functions u, v which satisfy the following system of elliptic equations

$$(1) \quad \begin{aligned} \varepsilon^2 \Delta u + f(u) - \kappa v &= 0, \\ D \Delta v + u - m - \gamma v &= 0, \end{aligned} \quad \text{in } \Omega.$$

Similarly the stationary solutions of $(E-2)_\varepsilon$ and $(E-3)_{\varepsilon,D}$ solve

$$(2) \quad \begin{aligned} \varepsilon^2 \Delta u + f(u) - \frac{\varepsilon}{\mu} v &= 0, \\ D \Delta v + u - m - \gamma v &= 0, \end{aligned} \quad \text{in } \Omega.$$

Note that these equations are independent of the constant τ . It is easy to see that if u, v solves (1), then u is a critical point of the functional I_ε defined by

$$I_\varepsilon[u] = \int_\Omega \frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) + \frac{D\kappa}{2} |\nabla(T(u-m))|^2 + \frac{\kappa\gamma}{2} \{T(u-m)\}^2 dx, \\ u \in H^1(\Omega),$$

where $T = (-D\Delta + \gamma)^{-1}$ is the Green operator of $-D\Delta + \gamma$ with the homogeneous Neumann boundary condition. We remark that if $\tau = 0$ were satisfied, the activator of $(E-1)_\varepsilon$, $u(\cdot, t)$ would be the gradient flow of I_ε . However since $\tau > 0$, the activator $u(\cdot, t)$ of $(E-1)_\varepsilon$ is different from a gradient flow of I_ε . In case of $(E-2)_\varepsilon$ and $(E-3)_{\varepsilon,D}$, we deal with the functionals J_ε and $J_{\varepsilon,D}$ respectively defined as follows:

$$J_{\varepsilon(D)}[u] = \int_\Omega \frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) + \frac{D\varepsilon}{2\mu} |\nabla(T(u-m))|^2 + \frac{\varepsilon\gamma}{2\mu} \{T(u-m)\}^2 dx.$$

(Note that the operator T depends on D .) It is easy to see that the family of the functionals I_ε and $J_{\varepsilon(D)}$ admit a global minimizer for each parameter. We are concerned with the asymptotic behavior of such minimizers for each parameter-scalings stated above. (For the stability, refer to [13].)

The homogenization problems with two length scales have been studied recently (refer to [1, 4, 9]). Also refer to [10, 12, 15] for the problem related to diblock copolymer.

We assume that f has polynomial growth at infinity and has three zeros: $-1, a, 1$ ($a \in (-1, 1)$) with $f'(\pm 1) < 0, f'(a) > 0$.

2 Statement of Main Results

To state the first result, we use the notion of Young measure, a useful tool for studying a sequence of functions which is oscillating and not convergent. We use the Young measure which is a map from Ω to the set of all probability measures on \mathbb{R} . A usual function $u(x)$ corresponds to the family of Dirac measures $\delta_{u(x)}$. The fundamental theorem for Young measure states the sufficient condition for relative compactness of a sequence of Young measures in an appropriate topology. We can get the limit Young measure instead of the limit function. (Refer to [14].)

In order to state the main result, define the constant

$$c_o = \frac{\sqrt{2}}{\int_{-1}^1 \sqrt{W(s)} ds}$$

and the set of all admissible functions in the limiting problem which we will obtain later,

$$\mathcal{G}(\Omega) = \{u \in BV(\Omega); |u(x)| = 1 \text{ for almost all } x \in \Omega\},$$

$$\mathcal{M}(\Omega) = \{u \in \mathcal{G}; \langle u \rangle_\Omega = m\}.$$

Here $\langle \cdot \rangle_\Omega$ denotes the average on Ω . We use the following notation: $P_\Omega(G)$ denotes the perimeter of $G \subset \Omega$ with respect to Ω .

Theorem 2.1. *The following statements hold:*

(i) *For any $\varepsilon > 0$, there exists a stable stationary solution $(u_\varepsilon, v_\varepsilon)$ of $(E-1)_\varepsilon$ such that for any sequence $\varepsilon_n \rightarrow 0$, u_{ε_n} is not convergent in $L^1(\Omega)$ and generates Young measure $\nu = (\nu_x)_{x \in \Omega}$ with $\nu_x = \frac{1-m}{2}\delta_{-1} + \frac{1+m}{2}\delta_1$ for almost all $x \in \Omega$.*

(ii) *For any sequence $\varepsilon_n \rightarrow 0$, there exists a subsequence $\varepsilon_k = \varepsilon_{n_k}$ and stable stationary solutions (u_k, v_k) of $(E-2)_{\varepsilon_k}$ such that u_k converges strongly in $L^1(\Omega)$ to a solution of*

$$(P)^\mu \quad \min_{u \in \mathcal{G}} B^\mu(u), \quad B^\mu(u) = \frac{2}{c_0} P_\Omega(\{u = 1\}) + \frac{1}{2\mu} \int_\Omega (u - m) T(u - m) dx.$$

(iii) *For any sequence $\varepsilon_n \rightarrow 0$, $D_n \rightarrow \infty$, there exist subsequences $\varepsilon_k = \varepsilon_{n_k}$, $D_k = D_{n_k}$ such that for each k , $(E-3)_{\varepsilon_k, D_k}$ has a stable stationary solution (u_k, v_k) which has the*

property that u_k converges strongly in $L^1(\Omega)$ to a solution of

$$(\tilde{P})^\mu \quad \min_{u \in \mathcal{G}} \tilde{B}(u), \quad \tilde{B}(u) = \frac{2}{c_0} P_\Omega(\{u = 1\}) + \frac{1}{2\mu\gamma} |\Omega| (\langle u \rangle - m)^2.$$

Note that the solutions in Theorem 2.1 (i) do not have a limit. In fact, from the result of [11], for $(E-1)_\varepsilon$, any stationary solutions which has a smooth surface as a limit must be unstable. In Theorem 2.1, we obtained the two limiting problems, $(P)^\mu$ and $(\tilde{P})^\mu$, which are the geometric minimization problem with a parameter dependence, and determine the location of interior boundary layers. The next theorem concerns the asymptotic behavior of solutions of the two problems $(P)^\mu$ and $(\tilde{P})^\mu$ as $\mu \rightarrow 0$.

Theorem 2.2. *The following statements hold:*

(i) Let u^μ be a solution of $(P)^\mu$. Then for any sequence $\mu_k \rightarrow 0$, u^μ generates the same Young measure ν as in Theorem 2.1 (i).

(ii) Let \tilde{u}^μ be a solution of $(\tilde{P})^\mu$. Then for any sequence $\mu_n \rightarrow 0$, there exists a subsequence $\mu_k = \mu_{n_k}$ such that \tilde{u}^{μ_k} converges strongly in $L^1(\Omega)$ to a solution u^* of

$$\min_{u \in \mathcal{M}} P_\Omega(\{u = 1\}),$$

and generates the Young measure $\nu = (\nu_x)_{x \in \Omega}$ with $\nu_x = \delta_{u^*(x)}$ for almost all $x \in \Omega$.

Note that for the problem $(P)^\mu$, we obtained a similar result as Theorem 2.1 (i), which corresponds to the case $\varepsilon = \mu\kappa$. We see that we can construct a sequence of solutions for $(E-2)_\varepsilon$ which converges to a pattern with an arbitrary large perimeter if we choose sufficiently small μ .

In the next Theorem, we derive the geometric interface equation associated with the solutions of $(P)^\mu$ and $(\tilde{P})^\mu$. We use the following notations: We take the sign of mean curvature such that principal curvature of the sphere is negative when the normal vector points to the center. ∂' denotes the relative boundary with respect to Ω .

Theorem 2.3. *The following statements hold:*

(i) For fixed $\mu > 0$, let u be a solution of $(P)^\mu$ and $\Gamma = \partial'\{u = 1\}$. Assume that Γ is smooth in a neighborhood U of a point $x_0 \in \Gamma$. Then there holds

$$\mu H = c_0 T(u - m), \quad \text{on } \Gamma \cap U,$$

where H denotes the mean curvature of Γ (when the normal vector points from $\{u = -1\}$ to $\{u = 1\}$).

(ii) For fixed $\mu > 0$, let \tilde{u} be a solution of $(\tilde{P})^\mu$ and $\tilde{\Gamma} = \partial'\{\tilde{u} = 1\}$. Assume that $\tilde{\Gamma}$ is smooth in a neighborhood \tilde{U} of a point $\tilde{x}_o \in \tilde{\Gamma}$. Then there holds

$$\mu H = \frac{c_o}{\gamma}(\langle \tilde{u} \rangle - m), \quad \text{on } \tilde{\Gamma} \cap \tilde{U},$$

where H denotes the mean curvature of $\tilde{\Gamma}$ (when the normal vector points from $\{\tilde{u} = -1\}$ to $\{\tilde{u} = 1\}$).

Theorem 2.3 (ii) implies that solutions of $(\tilde{P})^\mu$ typically involve a partition of Ω into regions separated by surfaces of a constant mean curvature. In [3], they obtained a limiting free boundary problem from an Allen–Cahn equation with a nonlocal term, which arises as a limit of a reaction–diffusion system. Then we see that any surface which corresponds to stationary solutions of the motion law obtained in [3] has also a constant mean curvature.

3 Remarks on Two Dimensional Problems

$u \in \mathcal{G}(\Omega)$ is called planar if $u = u(x_1, \dots, x_N)$, $(x_1, \dots, x_N) \in \Omega$ depends only on x_1 .

Proposition 3.1. *Let $N = 2$ and $\Omega = (0, 1)^2$. Then there exists a constant $m \in (-1, 1)$, sufficiently close to -1 , and a sequence $\mu_k \rightarrow 0$ such that every solution u^{μ_k} of $(P)^{\mu_k}$ is not planar.*

We think typical interfaces for solutions of $(\tilde{P})^\mu$ should be lines or circles when $N = 2$. We believe that, for sufficiently close to 1, and μ small, an interface approximated by a circle of a small radius, centered near the points on the boundary, which have the maximum mean curvature, should arise as in Cahn–Hilliard theory.

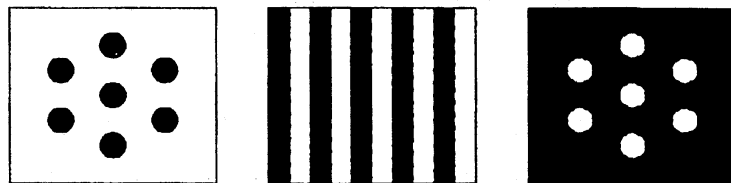


Figure 1 Typical Patterns; the black part is the region $u \sim 1$ and the white part is the region $u \sim -1$. (i) the left picture is the case $m < 0$; (ii) the central picture is the case $m \sim 0$; and the right picture is the case $m > 0$

We cannot expect that the minimizers of I_ε are precisely periodic in two dimensional arbitrary domain unlike the one dimensional case. However the Young measure generated by the global minimizers is constant in $x \in \Omega$. (See, Theorem 2.1 (i).) This suggests that the energy of global minimizers distribute somewhat uniformly. Then if the minimizers are not planar, what do they look like? In fact, non-planar minimizers which have hexagonal structures are observed (see Figure 1). We would like to give a mathematical account of this hexagonal pattern selection drawn in Figure 2.

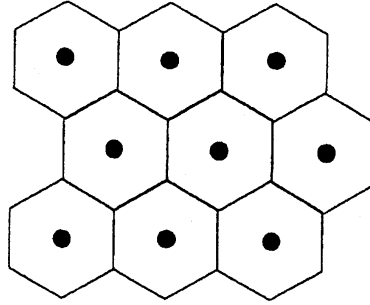


Figure 2 hexagon structure

Since the formal discussion suggests that we should study the pattern of the order $\varepsilon^{1/3}$, we use the following scaling and transformed functions

$$\begin{aligned}\hat{\varepsilon} &= \varepsilon^{2/3}, y = \frac{x}{\varepsilon^{1/3}}, \\ u(x) &= U(y), v(x) = \varepsilon^{2/3} V(y), \\ -D\Delta V + \gamma\hat{\varepsilon}V &= U - m.\end{aligned}$$

Now let U, V be extended to the whole \mathbb{R}^N in a symmetric and periodic way with a periodic unit domain Y . Then if $\{y; \hat{\varepsilon}y \in \Omega\}$ is packed with a finite number of translated Y , we have

$$\begin{aligned}\varepsilon^{-2/3}|\Omega|^{-1}I_\varepsilon[u] &= \\ \frac{1}{|Y|} \int_Y \frac{\hat{\varepsilon}}{2} |\nabla U|^2 + \frac{W(U)}{\hat{\varepsilon}} + \frac{(\langle U \rangle_Y - m)^2}{2\gamma\hat{\varepsilon}} + \frac{D\kappa}{2} |\nabla V|^2 + \frac{\kappa\gamma\hat{\varepsilon}}{2} (V - \langle V \rangle_Y)^2 dy.\end{aligned}$$

By using this rescaling argument and the Modica–Mortola theorem, we are led to the following reduced energy density:

$$\mathcal{E}[U] = \frac{1}{|Y|} \left[\frac{2}{c_0} P_Y(\{U = 1\}) + \frac{D\kappa}{2} \int_Y |\nabla V|^2 dy \right]$$

if U, V are Y -periodic functions such that $W(U) = 0$, $\langle U \rangle_Y = m$ and $-D\Delta V = U - m$. Then we get

$$I_\varepsilon[u] \sim |\Omega| \mathcal{E}[U] \varepsilon^{2/3}.$$

Note that the isoperimetric constant, the minimum of the perimeter with a volume constraint, is achieved if and only if the interface is the sphere. Now consider the dimension $N = 2$ and define the periodic circular patterns as follows. Let α, β be two complex numbers with $\text{Im}(\beta/\alpha) > 0$, and $\Sigma = \mathbb{Z}\alpha + \mathbb{Z}\beta$ be a lattice in the complex plane. Then let $U_\Sigma : \mathbb{R}^2 \rightarrow \{\pm 1\}$ be a function satisfying $U_\Sigma(x_1, x_2) = 1$ if $\text{dist}(x_1 + ix_2, \Sigma) \leq r$ and $U_\Sigma(x_1, x_2) = -1$ if $\text{dist}(x_1 + ix_2, \Sigma) > r$, with a constant $r > 0$ being determined by $\langle U \rangle_Y = m$. We assume that $r < \min\{|\alpha|, |\beta|\}$, which can be satisfied for a certain Σ if and only if $m \in (-1, \sqrt{3}\pi/3 - 1)$. Let $Y_\Sigma = \{(x_1, x_2); \text{there exist } s, t \in (0, 1) \text{ such that } x_1 + ix_2 = s\alpha + t\beta\}$ be a unit of parallelogram. See Figure 3.

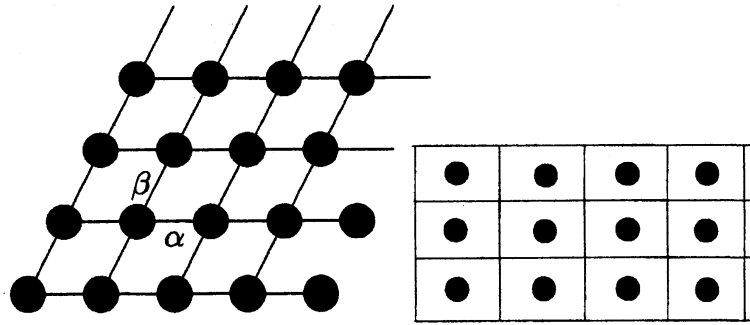


Figure 3 Periodic Circular Patterns U_Σ

One can show that the energy density for the triangle pattern (Figure 4) is larger than the hexagonal pattern (Figure 2).

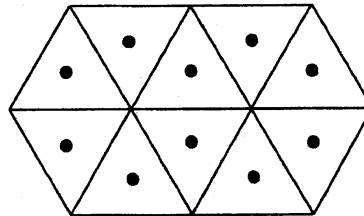


Figure 4 triangle structure

We will show that the energy density defined above achieves the minimum when Σ is a hexagonal structure. Then we will obtain the upper bound for the $\min I_\varepsilon$ for arbitrary domains by hexagonal structures, making a close study of the error by $\partial\Omega$.

Proposition 3.2 (X. Chen & Y. Oshita). *The following statements hold:*

(1) For $\Sigma = \mathbb{Z}\alpha + \mathbb{Z}\beta$, $\zeta = \beta/\alpha$, $\text{Im}(\zeta) > 0$,

$$\mathcal{E}[U_\Sigma] = \frac{2}{c_o} \sqrt{\frac{2\pi(1+m)}{|Y_\Sigma|}} + \frac{\kappa(1+m)^2 [R(\zeta) + c_1(m)] |Y_\Sigma|}{2D},$$

where

$$R(\zeta) = -\frac{1}{2\pi} \log \left| \sqrt{\text{Im}(\zeta)} q^{1/12} \prod_{n=1}^{\infty} (1 - q^n)^2 \right|, \quad q = e^{2\pi i \zeta}$$

and

$$c_1(m) = \frac{1}{4\pi} \left(1 + \frac{m}{2} - \log(2\pi(1+m)) \right).$$

(2) The minimum of $\mathcal{E}[U_\Sigma]$ among all possible periodic circular patterns is

$$\mathcal{E}^* = 3(1+m)(c_o)^{-2/3} D^{-1/3} [\pi \kappa(c_1(m) + R(\zeta^*))]^{1/3}, \quad \zeta^* = e^{i\pi/3},$$

which is attained when Σ is equal to the lattice $\mathbb{Z}\alpha^* + \mathbb{Z}\beta^*$,

$$|\alpha^*| = |\beta^*| = 2\pi^{1/6} 3^{-1/4} (1+m)^{-1/2} D^{1/3} [c_o \kappa(c_1(m) + R(\zeta^*))]^{-1/3}, \quad \frac{\beta^*}{\alpha^*} = \zeta^*.$$

(3) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the smooth boundary $\partial\Omega$. Then

$$\min_{u \in H^1(\Omega)} I_\varepsilon[u] \leq |\Omega| \varepsilon^{2/3} [\mathcal{E}^* + O(\varepsilon^{1/3} |\log \varepsilon|)],$$

as $\varepsilon \rightarrow 0$.

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